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An Approach to Estimation in Seismic Equalization

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Lexington, Massachusetts

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AN APPROACH TO ESTIMATION IN SEISMIC EQUALIZATION

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Group 64

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ABSTRACT

The "seismic equalization" problem is that of correcting the response at one station to match that at another station which may have different instrument characteristics and different (and unknown) local reverberation characteristics. In this note, the problem of seismic equalization is formulated mathematically, and that portion involving measurement or estimation of a transfer-function ratio is modeled and attacked on statistical terms, first by an ad hoc procedure and then by the method of maximum likelihood.

Accepted for the Air Force
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I. INTRODUCTION*

The statistical problem examined in this note is motivated by the need for "sensor equalization" in seismic array processing. To define "sensor equalization" we think of a sensor as a composite electromechanical transducer comprising (i) the local geology on which a seismic wave impinges, (ii) the coupling of this geology or terrain to the seismometer, and (iii) the seismometer itself. The "sensor response" (as the term will be used here) is the combined response of all these elements, that is, the impulse response or (frequency) transfer function that relates the signature of the wave to the seismometer electrical output of the sensor. Considering the nature of the elements (i) and (ii), it is only realistic to view the sensor responses as unknown, at least to some degree, and also to be a function of wave arrival angle.

The measurement of each sensor response is at best difficult, for we have no control over the waves that must be relied on to "probe," nor can we know their signatures in detail. The actual measurement of the complete sensor response is the basic "deconvolution" problem, which we will not discuss.

If, however, we address the more modest "equalization" goal of obtaining identical sensor outputs (in the absence of noise) to a wave arriving at a known angle without explicitly finding the sensor responses, then there is reason to be more hopeful. Of course, even should identical outputs be obtainable, they will still suffer some distortion of the wave signature--but this should not unduly disturb trained seismologists already familiar with such sensor aberrations.

Thus, in the equalization problem, we seek to determine the ratio between the complex-valued transfer functions $H_1(\omega)$ and $H_2(\omega)$ of a pair of sensors (this is equivalent to solving a linked pair of integral equations in the impulse responses, a conceptually more difficult task), and then to construct a filter having the transfer ratio $H_1(\omega)/H_2(\omega)$ [or $H_2(\omega)/H_1(\omega)$] as its transfer function. Such a filter (which if not happening to be

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* Seismic equalization has also been under study by Texas Instruments.[1] In addition, a paper by the late Dr. M. J. Levin[2] pertains to this problem.

physically realizable requires only that some delay be inserted in the system) when placed in tandem with the second sensor will convert its response into that of the first (or vice versa). In this note we confine attention to the measurement of $H_1(\omega)/H_2(\omega)$ rather than its realization, and deal with just a pair of sensors. Should there be more than two, we would equalize pair-by-pair, or perhaps equalize each with respect to the sum (previous to equalization) of all.

To perform the transfer-ratio measurement for a given arrival angle, we must rely on a collection of responses to probing waves (geophysically or atomically generated) that are of largely unknown signature but that are known to have arrived at that angle. Provided that for each event the same wave signature is received at each sensor input (apart from a fixed difference in relative amplitude) and that enough probing energy is accumulated at each frequency ω, relative to the system noise and to the number of responses in the collection, the measurement can be accomplished with vanishingly small rms error. If, however, the first condition is violated, as can happen for example if the source fault plane or radiation pattern (at a given frequency) varies from event to event and the seismometers are far apart, then equalization is probably unattainable. By means of the statistical analysis contained in this note it should be possible, through the setting up of confidence regions on the estimates of the transfer ratio from separate collections of responses, to test for such a contingency.

To enjoy the convenience of working in the frequency domain, we use as the collection of sensor outputs the Fourier transforms of a selected set of seismometer output waveform sections. The question arises of how long the output sections or observations should be--too short an observation results in smearing of spectral detail, while one that is too long contains an inordinate amount of noise. No guide on this point is presently available, other than intuition.

To be realistic, we must presume in general that there is correlation between the noises in the sensor outputs, and even that the correlation may vary from the arrival time of one wave to that of the next (sorting as to arrival angle may result in such infrequent probing that the noise field can change considerably). If not known, such

correlation can so bias the measurement of H_1/H_2 as to render it void. Fortunately, however, it is wholly reasonable to assume that the correlations, as well as the noise intensities at the two seismometer outputs, are known at all times. This of course requires additional data-processing beyond that implied by the development presented in the following Sections, but hardly more than is already employed at present in sophisticated array work.

We now proceed with a statistical analysis of the problem of measuring the ratio between a pair of transfer functions when noise disturbances are present.

II. THE STATISTICAL MODEL

We adopt the following model for the single-frequency equalization problem. (Equalization is to be accomplished frequency-by-frequency.) From data on N seismic events, we are given N pairs of Fourier-spectral observations $\{Y_{1i}, Y_{2i}\}$, known to be generated by

$$\left.\begin{aligned} Y_{1i} &= H_1 X_i + N_{1i} \\ Y_{2i} &= H_2 X_i + N_{2i} \end{aligned}\right\} \quad i = 1, \ldots, N \tag{1}$$

where the frequency parameter ω is henceforth left implicit. Here, the sensor transfer-functions H_1, H_2 and the event excitations (Fourier transforms of the wave signatures) $\{X_i\}$ are unknown complex constants (or mathematical, not random, variables) and the noise disturbances $\{N_{1i}\}$, $\{N_{2i}\}$ are zero-mean, complex gaussian variates, taken to be independent between observation pairs:

$$\overline{N_{ki} N_{\ell j}} = 0 = \overline{N_{ki}^* N_{\ell j}} \quad \text{when } i \neq j; \text{ for } k = 1,2 \text{ and } \ell = 1,2 \tag{2}$$

Our task is to form an estimate of the transfer-function ratio H_1/H_2 and to draw confidence regions about it, so that an effective signal equalizer may be constructed with the aim of converting the mean of any Y_{2i} into that of the corresponding Y_{1i}. In forming the estimate, we are allowed knowledge only of the $\{Y_{1i}, Y_{2i}\}$ and the noise statistics; in addition to the reasonable assumption (2), it is convenient to presuppose the observations to have been normalized (through individual weighting by positive real constants derived from the known noise intensities) so that

$$\overline{|N_{ki}|^2} = 2\,\overline{[\mathrm{Re}(N_{ki})]^2} = 2\,\overline{[\mathrm{Im}(N_{ki})]^2} = 1; \text{ all } i, k \tag{3}$$

Equation (3) implies statistical identity and independence between the real and imaginary components of each noise variate, conditions that in practice will as a rule

be met quite closely. The noise normalization both simplifies the analysis and has a valid basis in the well-accepted notion that one should play down observations known to have excessive noise and accent those for which the random disturbances are small. Furthermore, normalization is a reversible procedure, and no statistical "information" is lost in such an operation (in fact, the method of maximum likelihood can be shown to dictate such normalization).

In order to retain the basic character of our model (1) when normalization is applied, however, we require the further, and perhaps on occasion unrealistic, assumption

$$\sqrt{\overline{|N_{1i}|^2} \Big/ \overline{|N_{2i}|^2}} = \gamma > 0, \quad \text{a constant independent of } i \tag{4}$$

If $\gamma = 1$, the nature of the $\{X_i\}$ as unknowns leaves (1) quite unaltered after normalization; otherwise, the estimate of H_1/H_2 in the original, unnormalized situation is given by estimating H_1/H_2 from the normalized data and then multiplying by γ. [Should (4) be violated, our results would no longer apply, but an appropriate generalization of the analysis no doubt exists.]

Finally, we permit noise correlation to exist within any observation pair. The correlation must be known but may vary from pair to pair. This enables most seismic situations to be modeled realistically:

$$\overline{N_{1i}N_{2i}^*} = \rho_i \, ; \quad |\rho_i| < 1 \tag{5a}$$

We assume, however, that for all i

$$\overline{N_{1i}N_{2i}} = 0 \tag{5b}$$

That no complex correlation coefficient ρ_i can exceed unity in magnitude may be shown through the Schwarz inequality. The condition (5b) will be nearly true in most measurement situations.

Except that complex quantities are involved, our problem is not new; investigations of this type have been the subject of considerable post-war research among statisticians. Chapter 29 of Kendall and Stuart[3] furnishes an excellent introduction to this class of problems, showing that regression analysis constitutes a restricted subclass, and noting that the motivation and applications usually relate to the natural sciences and econometrics. (I am indebted to Dr. Max Halperin for this reference, which although treating a scalar rather than complex-valued situation, parallels and confirms my independent efforts at a number of points.)

In Kendall and Stuart ("K/S") terms, our problem is one of estimation within a "functional relation," where this relation is

$$H_1X_i = (H_1/H_2)\, H_2X_i \tag{6}$$

and H_1/H_2 is to be estimated when only randomly-perturbed versions of the $\{H_1X_i\}$ and $\{H_2X_i\}$ are observable, as in (1). In K/S, the scalar equivalents of the $\{H_1X_i\}$ and $\{H_2X_i\}$ are viewed either as non-random unknowns or as gaussian variates of unknown statistics, but both approaches yield similar results--in the seismic context it is probably more realistic to take the former view. Also in K/S, the statistics of the gaussian disturbances $\{N_{1i}\}$, $\{N_{2i}\}$ are considered unknown (but <u>not</u> i-dependent), and required to be estimated along with H_1/H_2 (again, however, we are interpreting their scalar results in terms of our notation). We assume, however, that these statistics are available (as is reasonable in the seismic context) even though the similarity between our results and those of K/S implies that this additional information may not actually be needed in formulating an estimator. (However, the confidence regions would be expected to differ according to whether the noise statistics are known or must be simultaneously estimated, and in fact we propose a different confidence procedure from that of K/S, one that draws on our presumed additional knowledge.)

III. AD HOC ESTIMATION

In first undertaking this study, we considered estimators of H_1/H_2 that were quite ad hoc, but which were completely explicit and fairly amenable to the statistical analysis required for setting confidence limits. We now present these early results, deferring to Section IV the generally more implicit, but often asymptotically more efficient, estimation procedure prescribed by the method of maximum likelihood.

Let us begin by momentarily imagining that the variances of the components of the $\{N_{1i}\}$, $\{N_{2i}\}$ are all zero, so that these variates vanish in (1), but that H_1, H_2 and the $\{X_i\}$ remain unknown. Then simply by forming the ratio Y_{1i}/Y_{2i}, for any i for which $Y_{2i} \neq 0$, the transfer ratio H_1/H_2 is immediately and exactly determined, since X_i cancels out. By this token, one next might attempt to average out the noise that is actually present through the estimator

$$\sum_{i=1}^{N} Y_{1i} \Bigg/ \sum_{i=1}^{N} Y_{2i} \tag{7}$$

True, the ratio between the numerator and denominator means is again exactly H_1/H_2, but such an estimate cannot be expected to do well in general. This is so because normally (at least in seismology) the X_i, as vectors, will lie in no constructive relationship, with the result that near-cancellation of the vector-sum mean component may occur in the numerator or the denominator of (7). When this happens, the estimate will be at the mercy of the noise.

What seems to be needed is an estimator whose performance depends on the magnitudes of the $\{X_i\}$, and not on their phase angles. This suggests that, while adhering to a ratio scheme so that the influence of the $\{X_i\}$ on the mean value of the estimate of H_1/H_2 may still be suppressed, we involve the observables in the numerator, and in the denominator, in a nonlinear way. The use of quadratics is particularly attractive, for much is known about the statistics of quadratic forms in gaussian variates, and we must look ahead to the need to obtain the estimate statistics, in order to determine confidence regions. There are a number of ways in which suitable

quadratics in the $\{Y_{1i}, Y_{2i}\}$ may be formed, and selection among them would at first seem to be largely a matter of taste. (Later, we shall show that maximum-likelihood estimation prescribes a particular selection of quadratics, and that the choice of quadratic-type nonlinearities themselves is no longer ad hoc in this context.)

We have taken the following path. First, we write the quantity to be estimated in polar form:

$$H_1/H_2 = Re^{j\theta} \; ; \; R > 0 \tag{8}$$

since there appears to be no preference a priori for estimating H_1/H_2 rather than H_2/H_1, and the polar form lends itself naturally to reciprocation. Next, we consider the estimation of R separately from that of estimating θ [either estimate will be seen to have a pleasing symmetry in the observables $\{Y_{1i}\}, \{Y_{2i}\}$ by virtue of the representation (8).] Our ad hoc estimate* of R is:

$$\tilde{R} = \left| \frac{\sum_{i=1}^{N} (|Y_{1i}|^2 - 1)}{\sum_{i=1}^{N} (|Y_{2i}|^2 - 1)} \right|^{1/2} \tag{9}$$

and of θ:

$$\tilde{\theta} = \tan^{-1} \left\{ \frac{\mathrm{Im}\left[\sum_{i=1}^{N} (Y_{1i}Y_{2i}^* - \rho_i)\right]}{\mathrm{Re}\left[\sum_{i=1}^{N} (Y_{1i}Y_{2i}^* - \rho_i)\right]} \right\} \tag{10}$$

with the quadrant assigned to $\tilde{\theta}$ according to the sign of either the numerator or denominator of (10).

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* It is interesting that the ad hoc estimator (9) is quite similar to a maximum-likelihood estimator (29.32) of K/S.

That both (9) and (10) are asymptotically unbiased and consistent estimates of R and θ, respectively, may be demonstrated by considering the typical term $(Y_{1i}Y_{2i}^* - \rho_i)$. [The terms in (9) may be viewed as special cases, in which $\rho_i = 1$ and H_1 replaces H_2 in (1), or vice versa.] We may write the term as

$$Y_{1i}Y_{2i}^* - \rho_i = \frac{1}{4}|Y_{1i} + Y_{2i}|^2 - \frac{1}{4}|Y_{1i} - Y_{2i}|^2 + \frac{j}{4}|Y_{1i} + jY_{2i}|^2 - \frac{j}{4}|Y_{1i} - jY_{2i}|^2 - \rho_i \tag{11}$$

and then examine a single squared magnitude at a time. Since later we will have need of a more general statistical treatment, let us actually find the mean value of a product $|Y_1|^2\,|Y_2|^2$ formed on the new variates

$$\left.\begin{aligned} Y_1 &= M_1 + N_1 \\ Y_2 &= M_2 + N_2 \end{aligned}\right\} \tag{12}$$

where M_1 and M_2 are given complex means and N_1, N_2 are zero-mean complex gaussian variates satisfying the conditions:

$$\left.\begin{aligned} \overline{N_k^2} = \overline{2\,[\mathrm{Re}(N_k)]^2} = \overline{2[\mathrm{Im}(N_k)]^2} = \sigma_k^2 \;;\; k = 1,2 \\ \overline{N_1N_2^*} = \rho_o\sigma_1\sigma_2 \;;\; \overline{N_1N_2} = 0 \end{aligned}\right\} \tag{13}$$

If we set $M_1 = M_2$, $\sigma_1 = \sigma_2$, and let $\rho_o = 0$, we have $\overline{|Y_1|^2\,|Y_2|^2} = \overline{|Y_1|^2}\;\overline{|Y_2|^2} = [\,\overline{|Y_1|^2}\,]^2$, while if $\rho_o = 1$, $\overline{|Y_1|^2\,|Y_2|^2} = \overline{|Y_1|^4} =$ variance $(|Y_1|^2) + [\,\overline{|Y_1|^2}\,]^2$. Thus we may study the mean and fluctuation of any of the squared magnitudes in (11) by taking the following general result (14), suitably identifying $M_1 = M_2$ with H_1X_i and H_2X_i through (1), setting the value of $\sigma_1 = \sigma_2$ by reference to (1) through (5b), and then either letting $\rho_o = 0$ or $\rho_o = 1$.

The general result is

$$\overline{|Y_1|^2\,|Y_2|^2} = \overline{[(m_{1R}+n_{1R})^2 + (m_{1I}+n_{1I})^2]\,[(m_{2R}+n_{2R})^2 + (m_{2I}+n_{2I})^2]}$$

$$= (m^2_{1R}+m^2_{1I}+\overline{n^2_{1R}}+\overline{n^2_{1I}})\,(m^2_{2R}+m^2_{2I}+\overline{n^2_{2R}}+\overline{n^2_{2I}})$$

$$+ \overline{(n^2_{1R}+n^2_{1I})\,(n^2_{2R}+n^2_{2I})} - (\overline{n^2_{1R}}+\overline{n^2_{1I}})\,(\overline{n^2_{2R}}+\overline{n^2_{2I}})$$

$$+ 4\,\overline{(m_{1R}n_{1R} + m_{1I}n_{1I})\,(m_{2R}n_{2R} + m_{2I}n_{2I})}$$

$$= (|M_1|^2 + \sigma_1^2)\,(|M_2|^2 + \sigma_2^2) + \sigma_1^2\,\sigma_2^2\,|\rho_o|^2 + 2\,\mathrm{Re}(M_1M_2^*\rho_o^*)\,\sigma_1\sigma_2 \tag{14}$$

where m_{kR}, m_{kI} and n_{kR}, n_{kI} are the real and imaginary parts of M_k and N_k, $k = 1,2$, respectively, and we have used [as a corollary of (13)],

$$\left.\begin{aligned} \overline{n_{1R}n_{2R}} &= \overline{n_{1I}n_{2I}} = \frac{\sigma_1\sigma_2}{2}\,\mathrm{Re}\,(\rho_o) \\ \overline{n_{1I}n_{2R}} &= -\,\overline{n_{1R}n_{2I}} = \frac{\sigma_1\sigma_2}{2}\,\mathrm{Im}\,(\rho_o) \end{aligned}\right\} \tag{15}$$

Also, we have invoked the fourth-moment relation for zero-mean gaussian random variables:

$$\overline{n_1n_2n_3n_4} = \overline{n_1n_2}\cdot\overline{n_3n_4} + \overline{n_1n_3}\cdot\overline{n_2n_4} + \overline{n_1n_4}\cdot\overline{n_2n_3} \tag{16}$$

Upon employing (14) in (11) with $\rho_o = 0$, $M_1 = M_2$, $\sigma_1 = \sigma_2$, we obtain for the mean:

$$\overline{Y_{1i}Y^*_{2i}} - \rho_i = \frac{1}{4}\,|X_i|^2\,[\,|H_1+H_2|^2 - |H_1-H_2|^2 + j\,|H_1+jH_2|^2 - j\,|H_1-jH_2|^2\,]$$

$$+ \frac{1}{2}\{[1+\mathrm{Re}(\rho_i)] - [1-\mathrm{Re}(\rho_i)] + j[1+\mathrm{Im}(\rho_i)] - j[1-\mathrm{Im}(\rho_i)]\} - \rho_i$$

$$= |X_i|^2\,H_1H_2^* \tag{17}$$

and with $\rho_o = 1$ we find for the variance of each of the squared magnitudes appearing in (11) a function of the form $C + D|X_i|^2$, where C and D depend on H_1, H_2 and ρ_i. Therefore, the variances of both the real and imaginary parts of (11) are upper-bounded by functions of this form, and further study shows that weaker but similar bounds can be obtained that do not involve ρ_i. Thus, the ratio of the variance of either the real or imaginary component of

$$\sum_{i=1}^{N} (Y_{1i}Y_{2i}^* - \rho_i)$$

to the squared mean of either component is upper-bounded by a function of the form

$$\frac{EN + F \sum_{i=1}^{N} |X_i|^2}{\left[\sum_{i=1}^{N} |X_i|^2\right]^2} \tag{18}$$

where E and F depend only on H_1 and H_2. Hence so long as

$$N^{-1/2} \sum_{i=1}^{N} |X_i|^2 \tag{19}$$

tends to infinity as $N \to \infty$ (it can be demonstrated that this is in general also a necessary condition), the ratio in (10) will converge (probabilistically) to $\mathrm{Im}(H_1H_2^*)/\mathrm{Re}(H_1H_2^*)$, so that this estimate of phase angle is asymptotically unbiased and consistent. (A little further work establishes that, asymptotically, the sign of the numerator or denominator will lie in the correct quadrant with probability one.)

By letting H_2 be replaced by H_1 and setting $\rho_i = 1$, we respectively force both the signals and the noises in (1) to be identical, so that $Y_{2i} = Y_{1i}$. Then by (17),

$$\overline{|Y_{1i}|^2} - 1 = |X_i|^2 |H_1|^2 \tag{20}$$

and similarly for $\overline{|Y_{2i}|^2}$. A variance analysis like that just sketched for the phase estimate (10) now shows that if again (19) tends to infinity with growing N, (9) is an asymptotically unbiased and consistent estimator of $R = |H_1/H_2|$. Thus, the _ad hoc_ estimators (9) and (10) together provide an asymptotically correct measurement of the transfer-function ratio H_1/H_2.

Let us next consider the problem of obtaining the statistics of (9) or (10) more precisely, now letting H_1, H_2 and the $\{X_i\}$ be known. As mentioned before, such analysis is needed in locating the confidence region about the estimate $\tilde{R}\, e^{j\tilde{\theta}}$ of H_1/H_2. [We consider (9) and (10) separately, although in principle the confidence region should be founded on their joint statistics.] The chief difficulty here is that of statistical dependence between numerator and denominator, which persists in (10) even when all the correlation coefficients $\{\rho_i\}$ vanish so that the $\{Y_{1i}, Y_{2i}\}$ become independent within as well as between pairs.

The first steps in handling the dependence are to write the cumulative probability function for the ratio in question in the form

$$\Pr\left(\frac{n}{d} < x\right) = \Pr(n\text{-}xd < 0,\ d > 0) + \Pr(n\text{-}xd > 0,\ d < 0);\ [\Pr(d = 0) = 0] \tag{21}$$

and then apply the bounds

$$\left.\begin{matrix} 1 - \Pr(n\text{-}xd > 0) - \Pr(d < 0) \\ \\ 0 \end{matrix}\right\} \le \Pr(n\text{-}xd < 0,\ d > 0) \le \left\{\begin{matrix} \Pr(n\text{-}xd < 0) \\ \\ \Pr(d > 0) \end{matrix}\right.$$

and similarly for $\Pr(n\text{-}xd > 0,\ d < 0)$. This yields

$$P(n\text{-}xd < 0) - \Pr(d < 0) \le \Pr\left(\frac{n}{d} < x\right) \le \Pr(n\text{-}xd < 0) + \Pr(d < 0) \tag{22}$$

and if attention is limited* to those situations in which (9) affords a reasonably good estimate of the magnitude R of H_1/H_2, Pr $(d < 0)$ will be entirely negligible. Turning to (10), good estimation conditions will not necessarily cause the denominator in the arctangent argument greatly to favor one sign, but should it not, then the numerator definitely will--therefore, in such circumstances the reciprocal of the argument can be tightly bounded,** and this is equally satisfactory.

With Pr $(\frac{n}{d} \le x)$ thus closely approximated by Pr $(n-xd < 0)$, a pair of linear transformations can be applied to the gaussian variates that appear quadratically in $(n-xd)$, to yield a new expression identical in sign to $(n-xd)$, but in which all the quadratics are now mutually independent. Proceeding in this manner for (9) (henceforth overlooking the approximation, and ignoring the magnitude signs in view of the virtual certainty that both numerator and denominator will be positive under good estimation conditions), we find

$$\Pr(\tilde{R} < r) = \Pr\left\{ \sum_{i=1}^{N} \left[|Z_{1i}|^2 - |Z_{2i}|^2 \right] / \left[8\,\mathrm{Re}(\mu_{Ri}) \right] - N(1-r^2) < 0 \right\} \tag{23}$$

where

$$\left.\begin{aligned} Z_{1i} &= Y_{1i}(1+\mu_{Ri}) + r(1-\mu_{Ri})\,Y_{2i} \\ Z_{2i} &= Y_{1i}(1-\mu_{Ri}^*) + r(1+\mu_{Ri}^*)\,Y_{2i} \end{aligned}\right\} \tag{24}$$

and

$$\mu_{Ri} = \frac{2jr\,\mathrm{Im}(\rho_i) + \sqrt{(1+r^2)^2 - 4r^2|\rho_i|^2}}{1 + r^2 - 2r\,\mathrm{Re}(\rho_i)} \tag{25}$$

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* In discussing their estimate (29.32), K/S cite the "inescapable difficulty" met with estimators of the type (9) – (10) when Pr $(d < 0) \ne 0$.

** Here, Pr $(d > 0)$ may just as likely be the negligible quantity, rather than Pr $(d < 0)$; in essence we are noting that under good estimation conditions there is virtually no chance of wrongly guessing the pair (or pairs) of adjacent quadrants in which the phase estimate will lie. K/S argue similarly in their Section 29.21.

For any i, it may be verified through (1), (3), and (5) that $\overline{Z_{1i}Z_{2i}^*} - (\overline{Z_{1i}} \cdot \overline{Z_{2i}^*}) = 0 = \overline{Z_{1i}Z_{2i}} - (\overline{Z_{1i}} \cdot \overline{Z_{2i}})$; this establishes the independence of Z_{1i} and Z_{2i}, since they are jointly gaussian variates. Since the real and imaginary parts of Z_{1i} or Z_{2i} are independent and of equal variance, the $\{|Z_{1i}|^2, |Z_{2i}|^2\}$ constitute a set of mutually independent non-central chi-square variates, each having two degrees of freedom.

It can be shown that if the noise correlations $\{\rho_i\}$ are all equal in magnitude ($|\rho_i| = |\rho|$ for all i, a condition that may not always apply in seismic equalization), the component variances of the $\{Z_{1i}\}$ are all equal, and likewise for the $\{Z_{2i}\}$. The sum

$$\sum_1 = \sum_{i=1}^{N} |Z_{1i}|^2 / [8\,\mathrm{Re}(\mu_{Ri})]$$

then has a non-central chi-square distribution of 2N degrees of freedom, with probability density given by

$$p_1(\textstyle\sum_1) = a_1^{-2}\,(\chi_1^2/\lambda_1)^{(N-1)/2}\,\exp[-(\chi_1^2/\lambda_1)/2]\; I_{N-1}(\sqrt{\chi_1^2\lambda_1});\ \textstyle\sum_1 > 0 \tag{26}$$

where $I_{N-1}(z)$ is the modified Bessel function of argument z and order (N-1). Here, $\chi_1^2 = \sum_1/a_1^2$ and

$$4a_1^2 = s + 1 - r^2\ ;\ s = \sqrt{(1+r^2)^2 - 4r^2|\rho|^2} \tag{27}$$

$$\lambda_1 = \frac{\sum_{i=1}^{N} |X_i|^2\,[\,|H_1|^2(1+r^2+s) + r^2|H_2|^2(1+r^2-s) - 4r^2\,\mathrm{Re}(H_1H_2^*\rho_i^*)\,]}{s^2 + s(1-r^2)} \tag{28}$$

A like result [the subscripts in (26) changing from "1" to "2"] obtains for the probability density of

$$\sum_2 = \sum_{i=1}^{N} |Z_{2i}|^2 / [8\,\mathrm{Re}(\mu_{Ri})]$$

with

$$4a_2^2 = s - 1 + r^2 \tag{29}$$

$$\lambda_2 = \frac{\sum_{i=1}^{N} |X_i|^2 \left[|H_1|^2 (1+r^2-s) + r^2 |H_2|^2 (1+r^2+s) - 4r^2 \operatorname{Re}(H_1 H_2^* \rho_i^*) \right]}{s^2 + s(1-r^2)} \tag{30}$$

Thus,

$$\Pr(\tilde{R} < r) = \int_0^\infty p_2(\textstyle\sum_2)\, d\textstyle\sum_2 \int_0^{\sum_2 + N(1-r^2)} p_1(\textstyle\sum_1)\, d\textstyle\sum_1 \tag{31}$$

and the statistics of the estimate (9) of $|H_1/H_2|$ are available through a double integral involving a pair of Bessel functions in the integrand. It does not appear possible to perform the integration analytically. Incidentally, when all the $\{\rho_i\}$ vanish, n and d in (21) involve quadratic variates that are independent, and it then happens that $\Pr(\tilde{R} < r)$ can be evaluated precisely through a combination of double integrations like that of (31) where the double inequalities in the probability statements of (21) are met through suitable choice of the integration limits. This affords a numerical check of the approximate, asymptotic result (38) that is given later.

The statistics of the phase estimate $\tilde{\theta}$ of (10) may be closely approximated in a manner paralleling that just developed for $\tilde{R}$. In this way we find, when $\rho_i = \rho$ for all i (equality of correlation magnitudes is no longer sufficient), and ignoring the approximation,

$$\Pr(\tan\tilde{\theta} < t) = \int_0^\infty p_4(\textstyle\sum_4)\, d\textstyle\sum_4 \int_0^{\sum_4 + N[\operatorname{Im}(\rho) - t\operatorname{Re}(\rho)]} p_3(\textstyle\sum_3)\, d\textstyle\sum_3 \tag{32}$$

if it is virtually certain that

$$\operatorname{Re}\left\{ \sum_{i=1}^{N} (Y_{1i} Y_{2i}^* - \rho) \right\} > 0 .$$

When the opposite is virtually certain, (32) equals $\Pr(\tan\tilde{\theta} > t)$; if neither is certain because the angle θ of H_1/H_2 lies near $(\pi/2)$ or $-(\pi/2)$, then we deal with ctn $\tilde{\theta}$ whose

denominator is of virtually certain sign, and obtain a similar expression to (32). In (32) p_3 and p_4 are given by (26) as re-subscripted, where

$$\left.\begin{aligned}
4a_3^2 &= u - \mathrm{Re}\,[(j+t)\rho]\,;\; u = \sqrt{1+s^2-\{\mathrm{Im}[(j+t)\rho]\}^2} \\
4a_4^2 &= u + \mathrm{Re}\,[(j+s)\rho] \\
\lambda_3 &= \frac{[\sum_{i=1}^{N} |X_i|^2]\,[(t^2+1)(|H_1|^2+|H_2|^2) - 2\mathrm{Im}[(j+t)H_1H_2^*]\,\mathrm{Im}[(j+t)\rho] - 2u\,\mathrm{Re}[(j+t)H_1H_2^*]}{4u^2 - 4u\,\mathrm{Re}\,[(j+t)\rho]} \\
\lambda_4 &= \frac{[\sum_{i=1}^{N} |X_i|^2]\,[(t^2+1)(|H_1|^2+|H_2|^2) - 2\mathrm{Im}[(j+t)H_1H_2^*]\,\mathrm{Im}[(j+t)\rho] + 2u\,\mathrm{Re}[(j+t)H_1H_2^*]}{4u^2 + 4u\,\mathrm{Re}\,[(j+t)\rho]}
\end{aligned}\right\} \tag{33}$$

(If all the $\{\rho_i\}$ vanish, (32) may be evaluated analytically through Price.[4]) Equations (32) and (33) follow from the relation, valid when the denominator of the arctangent argument in (10) is certain to be positive,

$$\Pr(\tan\tilde{\theta} < t) = \Pr\{\sum_{i=1}^{N} [\,|Z_{3i}|^2 - |Z_{4i}|^2]\,/\,[8\mathrm{Re}(\mu_{\theta i})] - \sum_{i=1}^{N} [\mathrm{Im}(\rho_i) - t\mathrm{Re}(\rho_i)] < 0\} \tag{34}$$

Here the $\{Z_{3i}, Z_{4i}\}$ are mutually independent gaussian variates given by

$$\left.\begin{aligned}
Z_{3i} &= [Y_{2i} - (\tfrac{j+t}{4})\,Y_{1i}]\,(1+\mu_{\theta i}) + [Y_{2i} + (\tfrac{j+t}{4})\,Y_{1i}]\,(1-\mu_{\theta i}) \\
Z_{4i} &= [Y_{2i} - (\tfrac{j+t}{4})\,Y_{1i}]\,(1-\mu_{\theta i}^*) + [Y_{2i} + (\tfrac{j+t}{4})\,Y_{1i}]\,(1+\mu_{\theta i}^*)
\end{aligned}\right\} \tag{35}$$

where

$$\mu_{\theta i} = \frac{-4j\,\mathrm{Im}\,[(j+t)\rho_i] + 4\sqrt{1+t^2-\{\mathrm{Im}\,[(j+t)\rho_i]\}^2}}{1+t^2} \tag{36}$$

Thus, when all the noise correlations $\{\rho_i\}$ are equal and good estimation conditions exist, we are able to obtain from (26) through (33) quite accurate results for the statistics of the estimated magnitude and phase of the transfer-function ratio H_1/H_2. Using these statistics, confidence regions can be set up about the estimate of H_1/H_2, and by treating different sets of observation-pairs the validity of the model (1) itself can be quantitatively tested. It is interesting to note that, as desired, the phase angles of the $\{X_i\}$ never affect the performance of the estimate; moreover, when all the $\{\rho_i\}$ are equal, the performance does not depend on the individual $\{|X_i|\}$ but only on the total signal "energy" in the observations

$$E = \sum_{i=1}^{N} |X_i|^2 \tag{37}$$

as multiplied by either one of the energy gains $|H_1|^2$, $|H_2|^2$. This is fortunate, for it leaves just a single unspecified parameter in the determination of the confidence region (assuming that a confidence percentage has been assigned and that there is no question of how this probability should be distributed in the excluded region--see Chapter 20 of K/S). Increasing this parameter E (presumably) shrinks the confidence region, and E can probably be conservatively estimated through the numerator or denominator of (9), or perhaps both together.

A complicating factor in determining the confidence region is that one does not have the joint statistics of $\tilde{R}$ and $\tilde{\theta}$ (they are generally dependent, even for asymptotically large N) and yet in general their individual statistics depend on both true values R and θ. This problem needs further examination (or recourse to a search of the literature), but the following expedient seems reasonable. First, for the estimate $\tilde{R}$ given by (9) find the confidence intervals for R corresponding to all values of the true parameter θ, using (26) through (31) (if all the $\{\rho_i\}$ vanish, there will be no dependence on θ). Then do the converse for the estimate $\tilde{\theta}$ given by (10), using (32) – (33). In this manner we obtain two regions, both in R and θ. The intersection

of these two regions may be taken to be the final confidence region, with a confidence of the order of whatever common percentage was adopted in setting up all of the confidence intervals.

It seems plausible that as $N \to \infty$, the quadratic sums in the right members of (23) and (34) will become gaussianly distributed. At least, this occurs if all the noise correlations $\{\rho_i\}$ are equal, for then the component variances of the Z_{1i}, Z_{2i}, Z_{3i}, and Z_{4i} do not depend on i, and the $|X_i|^2$ enter into the statistics of the sums only through their sum. Thus when the latter sum is imagined to be reapportioned equally among all i, the Central Limit Theorem applies to each of the quadratic sums. The range of the $\{\rho_i\}$ being restricted ($|\rho_i| \leq 1$), one feels that $N \to \infty$ implies sufficient "bunching" of the $\{\rho_i\}$ that the Central Limit Theorem is still effective, although this certainly remains to be proven.

If, furthermore, estimation conditions improve as $N \to \infty$, as they will if the total signal energy E of (37) grows faster than $\sqrt{N}$, then in (23) and (34) the local linearity of the right members in r and $\tan^{-1} t$, respectively, implies that $\tilde{R}$ and $\tilde{\theta}$ will themselves be asymptotically gaussian (with means equaling the true values R and θ). Here, we are drawing on the asymptotic consistency and lack of bias established earlier for (9) and (10). It is of interest to determine the variances of $\tilde{R}$ and $\tilde{\theta}$ in the asymptotic situation, for if they are asymptotically normal these are all that we need in order to draw a confidence region. By employing (14) with $\rho_o = 0$ or 1 in (23) and (34), some manipulation shows that the asymptotic variances are given by

$$\frac{\sigma_R^2}{R^2} = \frac{\sum_{i=1}^{N} |X_i|^2 \left[R + R^{-1} - 2|\rho_i| \cos(\theta - \theta_i)\right]}{2|H_1|\,|H_2|\,E^2} + \frac{\sum_{i=1}^{N} \left[R^2 + R^{-2} - 2|\rho_i|^2\right]}{4\,|H_1|^2\,|H_2|^2\,E^2} \tag{38}$$

$$\sigma_\theta^2 = \frac{\sum_{i=1}^{N} |X_i|^2 \left[R + R^{-1} - 2|\rho_i| \cos(\theta - \theta_i)\right]}{2\,|H_1|\,|H_2|\,E^2} + \frac{\sum_{i=1}^{N} \left[1 - |\rho_i|^2 + 2|\rho_i| \cos^2(\theta - \theta_i)\right]}{2\,|H_1|^2|H_2|^2\,E^2} \tag{39}$$

where $\rho_i = |\rho_i| \exp[j\theta_i]$ and E is given by (37). [That the asymptotic means are R and θ is verified concurrently. These mean-and-variance results can no doubt be derived directly from (9) and (10) and the assumption of improving estimation conditions ($E \to \infty$), without any appeal to asymptotic normality, and perhaps even without requiring $N \to \infty$.]

IV. Maximum-Likelihood ("ML") Estimation

We now take an entirely synthetic rather than *ad hoc* approach to the problem of estimating the transfer-function ratio H_1/H_2 from the given complex-valued observation pairs $\{Y_{1i}, Y_{2i}\}$, $i = 1, \ldots, N$, generated as in (1). Setting up the likelihood function, i.e., the probability density function of the $\{Y_{1i}, Y_{2i}\}$, we proceed to choose the unknowns H_1, H_2 and $\{X_i\}$ so that this function achieves its maximum value, and take the ratio H_1/H_2 thus found as our estimate $\widehat{H_1/H_2}$. Of course, the notion that it is good to maximize the likelihood is *ad hoc* in the first place. Furthermore, we shall see that this estimate is in general far less explicit than (9) – (10) and even when explicit, its statistics are usually difficult to derive. When estimation conditions are sufficiently good, however, the ML estimate can be definitely superior to the *ad hoc* estimate provided by (9) – (10) and thus certainly merits our attention.

We begin as before by assuming the conditions (2), (3), and (5). Should the noise intensities initially differ from observation to observation, but (4) be satisfied, it is not difficult to show that the ML method dictates the noise normalization as given. Rather than deal with the $\{Y_{1i}, Y_{2i}\}$ directly, which are usually correlated, we may equivalently and more conveniently maximize the likelihood of the linearly transformed (and still gaussian) variates

$$
\begin{aligned}
Y'_{1i} &= [Y_{1i} + (\sqrt{1-\rho_{Ii}^2} - j\rho_{Ii})\, Y_{2i}] \,/\, \sqrt{1-\rho_{Ii}^2 + \rho_{Ri}\sqrt{1-\rho_{Ii}^2}} \\
Y'_{2i} &= [(-\sqrt{1-\rho_{Ii}^2} + j\rho_{Ii})\, Y_{1i} + Y_{2i}] \,/\, \sqrt{1-\rho_{Ii}^2 - \rho_{Ri}\sqrt{1-\rho_{Ii}^2}}
\end{aligned}
\qquad (40)
$$

where

$$\rho_i = \rho_{Ri} + j\rho_{Ii} \qquad (41)$$

is the known noise correlation coefficient for the i^{th} observation pair. A little calculation now shows that the $\{Y'_{1i}, Y'_{2i}\}$ are mutually independent, and have real and imaginary components all of equal variance.

Upon finding the means of the $\{Y'_{1i}, Y'_{2i}\}$ from (1), we determine that the likelihood is maximized when we minimize the quantity

$$\sum_{i=1}^{N} \frac{\left|[Y_{1i} + Y_{2i}(\sqrt{1-\rho_{Ii}^2} - j\rho_{1i})] - X_i[H_1 + H_2(\sqrt{1-\rho_{Ii}^2} - j\rho_{Ii})\right|^2}{1-\rho_{Ii}^2 + \rho_{Ri}\sqrt{1-\rho_{Ii}^2}}$$

$$+ \sum_{i=1}^{N} \frac{\left|[Y_{1i}(-\sqrt{1-\rho_{Ii}^2} + j\rho_{Ii}) + Y_{2i}] - X_i[H_1(-\sqrt{1-\rho_{Ii}^2} + j\rho_{Ii}) + H_2]\right|^2}{1-\rho_{Ii}^2 - \rho_{Ri}\sqrt{1-\rho_{Ii}^2}} \tag{42}$$

This is most conveniently done term-by-term, maximizing on the $\{X_i\}$ while at first holding H_1 and H_2 fixed. Expanding the squared magnitudes, it is seen by inspection that for $|X_i|$ given, the minimizing angle of X_i is that of $[H_1^*Y_{1i}+H_2^*Y_{2i}-\rho_{Ri}(H_1^*Y_{2i}+H_2^*Y_{1i}) - 2j\rho_{Ii}(H_1^*Y_{2i}-H_2^*Y_{1i})]$. The quadratic in $|X_i|$ obtained upon substituting this result back in the i^{th} term of the expanded version of (42) may now be straightforwardly minimized by differentiation, whereupon our ML estimate of X_i is found to be

$$\hat{X}_i = \frac{H_1^*Y_{1i} + H_2^*Y_{2i} - \rho_i H_1^*Y_{2i} - \rho_i^* H_2^* Y_1}{|H_1|^2 + |H_2|^2 - 2\,\mathrm{Re}\,(H_1 H_2^* \rho_i^*)} \tag{43}$$

With (43) substituted in (42) we determine, after considerable algebra, that now

$$\sum_{i=1}^{N} \frac{|Y_{1i} - (H_1/H_2)\,Y_{2i}|^2}{1 + |H_1/H_2|^2 - 2\,\mathrm{Re}\,[(H_1/H_2)\,\rho_i^*]} \tag{44}$$

is to be minimized with respect to H_1 and H_2. Note, however, that (44) involves H_1 and H_2 only through H_1/H_2, the very parameter that we seek to estimate. In general,

we must stop here, leaving it to a computer to perform the minimization of (44) with respect to the magnitude and angle of H_1/H_2. Even when all the $\{\rho_i\}$ are equal but non-zero, it does not seem possible to proceed explicitly beyond the maximization with respect to magnitude for a given angle, or vice versa.

When the $\{\rho_i\}$ all vanish, however, we find that the minimizing angle $\hat{\theta}$ is that of

$$\sum_{i=1}^{N} Y_{1i} Y_{2i}^*$$

so that in this case the ad hoc (10) and ML estimates are identical. On the other hand, the ML estimate of the transfer-function magnitude is found to be, for vanishing noise correlations,

$$\hat{R} = \frac{\sum_{i=1}^{N} [\,|Y_{1i}|^2 - |Y_{2i}|^2\,]}{2\,|\sum_{i=1}^{N} Y_{1i} Y_{2i}^*|} + \sqrt{\left(\frac{\sum_{i=1}^{N} [\,|Y_{1i}|^2 - |Y_{2i}|^2\,]}{2\,|\sum_{i=1}^{N} Y_{1i} Y_{2i}^*|}\right)^2 + 1} \tag{45}$$

which is quite in contrast to (9), even though quadratics in the $\{Y_{1i}, Y_{2i}\}$ are involved in both. [Equation (45) does not appear amenable to statistical analysis except when

$$|\sum_{i=1}^{N} Y_{1i} Y_{2i}^*|$$

can with high probability be closely approximated by Re $\{\sum_{i=1}^{N} Y_{1i} Y_{2i}^* e^{-j\theta}\}$,

where θ is the angle of H_1/H_2.] It is interesting to compare (45) with the similar result (29.29) that K/S obtain in the scalar analog to (1).

The final goal of this analysis is to study the asymptotic behavior of the estimate $\widehat{H_1/H_2}$ that maximizes (44), and to compare it with that of the ad hoc estimate provided by (9) – (10). It is fortunate that even though the ML estimate itself must in general be found through trial-and-error, its asymptotic statistics can be determined

quite explicitly. Our procedure is to evaluate the second derivative of the mean value of (44), taken with respect to the logarithm of the magnitude (or the phase) of H_1/H_2 at the true value of H_1/H_2 (where the mean of (44) must asymptotically have its maximum), and divide the square of this derivative into the variance of the companion first derivative. The result is the asymptotic variance of the log-magnitude (or phase) estimate.

Performing this analysis for the log-magnitude $\beta = \log R$, where R is given by (8) we find by applying (14) to the derivatives of (44) and doing considerable algebra that the asymptotic variance of the ML estimate is (since $\overline{\hat{\beta}} = \beta$)

$$\overline{(\hat{\beta} - \beta)^2} = \frac{1}{2}\left[\sum_{i=1}^{N} \frac{|X_i|^2 |H_1| |H_2|}{R + R^{-1} - 2|\rho_i| \cos(\theta - \theta_i)}\right]^{-1}$$

$$+ \frac{1}{2}\left[\sum_{i=1}^{N} \frac{1 - |\rho_i|^2}{[R + R^{-1} - 2|\rho_i| \cos(\theta - \theta_i)]^2}\right]\left[\sum_{i=1}^{N} \frac{|X_i|^2 |H_1| |H_2|}{R + R^{-1} - 2|\rho_i| \cos(\theta - \theta_i)}\right]^{-2} \quad (46)$$

where again θ_i is the angle of the i^{th} noise correlation coefficient. Upon carrying through the like analysis for the asymptotic variance of the ML phase estimate $\hat{\theta}$, we obtain identically the same result as (46), which is quite pleasing, and to be contrasted with (38) and (39) for the *ad hoc* estimation. Moreover, the magnitude and phase errors of the *ad hoc* estimates are generally found to be coupled, whereas further analysis shows that the first derivative of (44) with respect to β is asymptotically uncorrelated with that with respect to θ, when both are taken at the true value of $H_1/H_2 = \exp(\beta + j\theta)$.

Thus, the errors in the ML estimate are asymptotically uncorrelated, and since the estimate should have gaussian statistics asymptotically (we beg this question), the errors are asymptotically independent. An asymptotically circular confidence region, centered on the estimate, can therefore be drawn in the plane whose rectangular

coordinates are (β, θ), where the radius is determined by the desired confidence and is proportional to the square root of (46). [The asymptotic independence of the ML log-magnitude and phase estimates, and their common asymptotic variance, imply that the real and imaginary parts of H_1/H_2 have asymptotic statistical properties like those of (β, θ).]

We close this study with a comparison of (46) against (38) and (39), to see how the asymptotic performance of the ML estimate relates to that of the ad hoc estimate (9) – (10). The first term of (46) may be shown through the Schwarz inequality to never exceed either of the (common) first terms of (38) and (39), while if the $\{|\rho_i| \cos(\theta - \theta_i)\}$ happen to be the same for all i, the second term of (46) is by inspection less than or equal to either of the second terms of (38) and (39). (Note: $R^2 + R^{-2} \geq 2$ for all positive R; also, when all the $\{\rho_i\}$ vanish, (39) equals (46), as it should since (10) then happens to be the ML estimator.) Thus, under these circumstances the ML estimate is uniformly better* than the ad hoc estimate, at least asymptotically.

If, however, the $\{|\rho_i| \cos(\theta - \theta_i)\}$ (which are the projections of the complex noise correlations on the H_1/H_2 vector) are permitted to depend on i (this is the actual seismic situation), the ML estimate can be poorer than the ad hoc, and to an unlimited degree (while both nonetheless remain in their asymptotic regions). To illustrate, let us suppose that $N = 2$, $R = |H_1/H_2| = 1$, $\theta_1 = \theta_2 = \theta$, $X_1 = 0 = \rho_2$. Then for the variance of the ML estimate of either the log-magnitude or the phase, we have from (46)

---

* It cannot be hoped that the ML estimate is uniformly better than all other estimates, even asymptotically, because of the presence of the unknown "incidental" parameters $\{X_i\}$--certainly, other estimators exist that will, by chance, be better "tuned" to a particular set of $\{X_i\}$ than the ML estimator. For example, the ML procedure uses all observations, irrespective of whether they actually contain probing energy or not; other estimators may fortuitously reject such observations. Perhaps the ML estimator is best in some minimax sense, but this remains to be shown--for studies of the effects of incidental parameters, see the papers by Neyman and Scott, and by Kiefer and Wolfowitz, referenced by K/S.

$$\overline{(\hat{\beta} - \beta)^2} = |X_2H_1|^{-2} [1 + |X_2H_1|^{-2}/(1 - |\rho_1|)]$$

while for the ad hoc estimates we have from (38) – (39)

$$\sigma_R^2/R^2 = |X_2H_1|^{-2} [1 + |X_2H_1|^{-2} (1 - \frac{|\rho_1|}{2})]$$

$$\sigma_\theta^2 = |X_2H_1|^{-2} [1 + |X_2H_1|^{-2} (1 + |\rho_1| - \frac{|\rho_1|^2}{2})]$$

As $|\rho_1| \to 1$, $|X_2H_1|$ must clearly be increased much more for ML estimation than for ad hoc estimation, to obtain comparable performance (the value of $|X_2H_1|$ being great enough that the variances are asymptotically small in either case). Of course, were we somehow to know that $X_1 = 0$ in performing the estimation, we would naturally ignore the first pair of observations, then obtaining $|X_2H_1|^{-2} [1 + |X_2H_1|^{-2}/2]$ as the common variance for the log-magnitude and phase estimation by both the ML and ad hoc methods. This, however, would be clairvoyance.

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13. ABSTRACT

The "seismic equalization" problem is that of correcting the response at one station to match that at another station which may have different instrument characteristics and different (and unknown) local reverberation characteristics. In this note, the problem of seismic equalization is formulated mathematically, and that portion involving measurement or estimation of a transfer-function ratio is modeled and attacked on statistical terms, first by an ad hoc procedure and then by the method of maximum likelihood.

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